Łukasiewicz logic as a logic of vague propositions: A short excursion

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Vaguely Speaking

(A prologue)

Jan Łukasiewicz 1878–1956

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- *…*
- *If* (10^{100}-*1*) *grains of wheat do not make a heap, then* 10^{100} *do not.*
- *Hence:* 10^{100} *grains of wheat do not make a heap.*

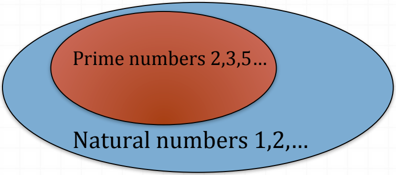
The Sorites Paradox

- Modern response: *Theories of Vagueness.*
- Initial problem: the monadic predicate `Heap(x)` is vague.
- To explain the paradox away we need a theory of such vague predicates.
- Any such theory needs some pre-theoretical, or at least theory-neutral, understanding of what a "vague predicate" is.
- Building on such a common pre-theoretical understanding of vagueness, a plethora of conflicting theories of vagueness has been advanced in the 20th century.
- So there is no explanation of the Sorites Paradox that is "standard", in the sense of being most widely accepted.

Theory-neutral features of vagueness

Features of a precise predicate.

The monadic predicate $P(x) :=$“x is prime”, interpreted over the set of natural numbers $x \geq 1$, is (absolutely) *precise*: its extension is the set of prime numbers; its anti-extension is the set of composite numbers; each number either belongs to the extension of P or to its anti-extension, but not to both; and in principle there is no issue as to whether a given number be prime or composite — though in practice it may be impossible to ascertain which is the case for an astronomic instance of x.



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By contrast, the monadic predicate $R(x) :=$"x is red", interpreted over the set of all objects, is (to some extent) *vague*: its extension ought to be the set of all red objects; its anti-extension ought to be the set of all non-red objects; but it may not be clear, even in principle, just which objects do qualify as red, and which as non-red — think of a peculiar tint at the borderline between red and pink.

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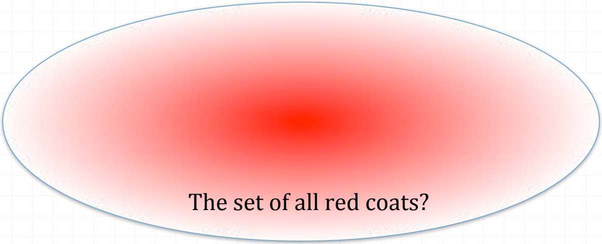

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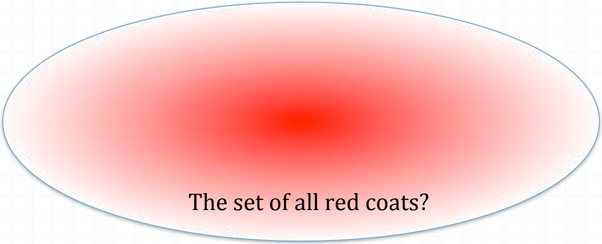

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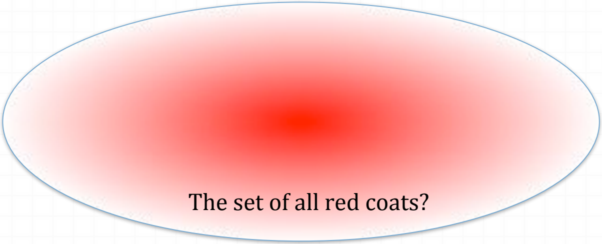


Theory-neutral features of vagueness

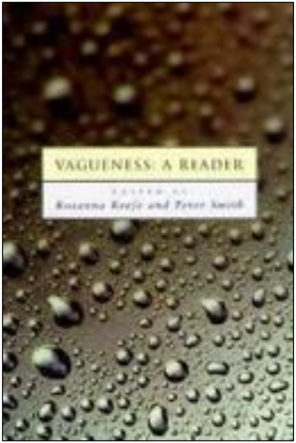
Features of a (monadic) vague predicate *R*:

(FV1) R admits *borderline cases* over the intended domain of interpretation D, *i.e.* there are instantiations of $R(x)$ by (a term naming a constant) $c \in D$ such that it is unclear whether $R(c)$ holds or its negation $\neg R(c)$ does.

(FV2) R lacks *sharp boundaries* over the intended domain of interpretation D, *i.e.* there is no clearly defined boundary separating the extension of $R(\cdot)$ from its anti-extension.

(FV3) R is susceptible to a *Sorites series* over the intended domain of interpretation D, *i.e.* there are instantiations of $R(x)$ by $c_1, \ldots, c_n \in D$ such that it is clear that $R(c_1)$ holds, it is clear that $R(c_n)$ does not hold, and it seems at least plausible that if $R(c_i)$ holds then so does $R(c_{i+1})$, for each $i \in \{1, \ldots, n-1\}$.

Theories of vagueness



Useful Reader: R. Keefe and P. Smith, eds.

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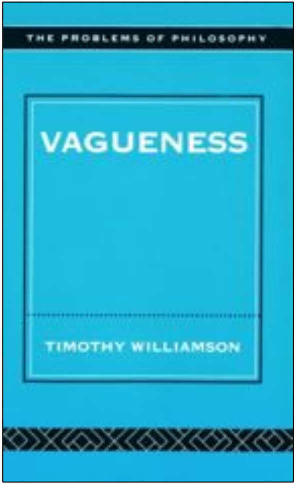
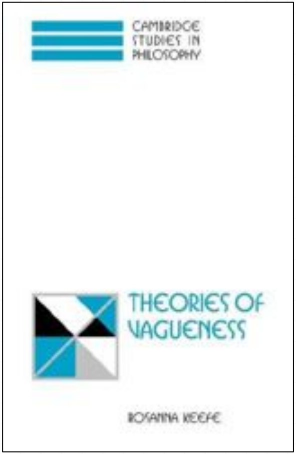


Epistemicism: Vagueness as Ignorance

Theories of vagueness



Supervaluationism: Vagueness as Precisifiability

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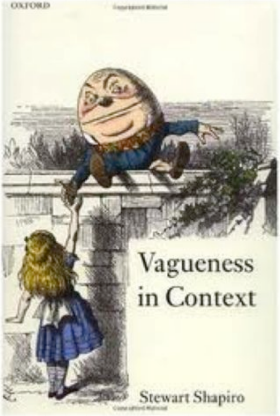
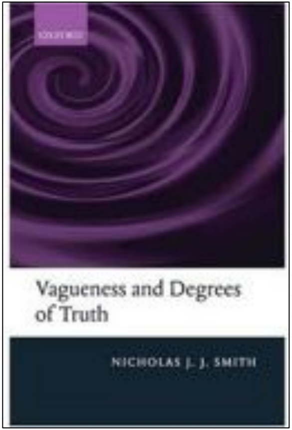


Contextualism: Vagueness as dependence from Context

Theories of vagueness



Degree-Based Theories: Vagueness as Truth-in-Degrees

Degree-Based Theories of Vagueness

Main Assumption: Truth comes in degrees.

- If x is a clear case of R, then $R(x)$ is (fully, classically) true.
- If x is a clear non-case of R, then $R(x)$ is (fully, classically) false.
- If x is a borderline case of R, then $R(x)$ is true (or false) **to a degree**.

It may seem natural to say that, in borderline cases, a certain coat is neither clearly red, nor clearly non-red, so that "*This coat is red*" is **neither true nor false**. And the further step of then saying that "*This coat is red*" is true (or false) **to some degree** may also sound appealing. (Well, does it sound appealing to you?) But we should be aware that taking this direction is a **major departure from the roots of logic** as we know it, both philosophically and mathematically.

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Frege on Truth

We are therefore driven into accepting the truth value [*Wahrheitswert*] of a sentence as constituting its reference [*Bedeutung*]. By the truth value of a sentence I understand the circumstance that it is true or false. There are no further truth values. For brevity I call the one the True [*das Wahre*], the other the False [*das Falsche*].

G. Frege, *On Sense and Reference*, 1892, p. 34.

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In other writings (notably the unpublished *Logik*), Frege makes the following very clear.

- Truth is a primitive notion in logic: it cannot be defined.
- `True(`p`)` is a peculiar predicate in that it does not admit comparatives: p is truer than q is a *façon de parler* lacking genuine logical content.
- (Implicitly.) In particular, **degrees of truth are non-sense**, according to Fregean orthodoxy.

Main objections to degree-theoretic accounts of vagueness:

1. Compositionality (K. Fine, 1975).
2. Higher-order vagueness (T. Williamson *et al.*, 1994).
3. Artificial precision (R. Keefe *et al.*, 2000).

I ignore higher-order vagueness here. I report Fine's arguments against compositionality, and those of Keefe *et al.* on artificial precision. For further information on artificial precision and related issues:

- V.M., *The problem of artificial precision in theories of vagueness: the rôle of maximal consistency*, Erkenntnis, to appear.
- V.M., *Is there a probability theory of many-valued events?*, in Probability, uncertainty and rationality, CRM Series, 10, Ed. della Normale, Pisa, 2010.

K. Fine, *Vagueness, Truth, and Logic*, Synthese, 1975.

KIT FINE

VAGUENESS, TRUTH AND LOGIC[1]

This paper began with the question 'What is the correct logic of vagueness?' This led to the further question 'What are the correct truth-conditions for a vague language?', which led, in its turn, to a more general consideration of meaning and existence. The first half of the paper contains the basic material. Section 1 expounds and criticizes one approach to the problem of truth-conditions. It is based upon an extension of the standard truth-tables and falls foul of something called penumbral connection. Section 2 introduces an alternative framework, within which penumbral connection can be accommodated. The key idea is to consider not only the truth-values that sentences actually receive but also the truth-values that they might receive under different ways of making them more precise. Section 3 describes and defends the favoured account within this framework. Very roughly, it says that a vague sentence is true if and only if it is true for all ways of making it completely precise. The second half of the paper deals with consequences, complications and comparisons. Sec-

Is any account along truth-value lines acceptable? Any account that satisfies the conditions F and S would always appear to make correct allocations of definite truth-value. However, even the maximizing policy fails to make many correct allocations of definite truth-value. For suppose that a certain blob is on the border of pink and red and let *P* be the sentence 'the blob is pink' and *R* the sentence 'the blob is red'. Then the conjunction *P* & *R* is false since the predicates 'is pink' and 'is red' are contraries. But on the maximizing account the conjunction *P* & *R* is indefinite since both of the conjuncts *P* and *R* are indefinite.

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The specific examples chosen should not blind us to the general point that they illustrate. It is that logical relations may hold among predicates with borderline cases or, more generally, among indefinite sentences. Given the predicate 'is red', one can understand the predicate 'is non-red' to be its contradictory: the boundary of the one shifts, as it were, with the boundary of the other. Indeed, it is not even clear that convincing examples require special predicates. Surely $P \& -P$ is false even though *P* is indefinite.

Let us refer to the possibility that logical relations hold among indefinite sentences as *penumbral connection*; and let us call the truths that arise, wholly or in part, from penumbral connection, *truths on a penumbra* or *penumbral truths*. Then our argument is that no natural truth-value approach respects penumbral truths. In particular, such an approach cannot distinguish between 'red' and 'pink' as independent and as exclusive upon their common penumbra.

Fine's argument about penumbral connection is considered by most as a definitive objection to any attempt of regarding a (truth-functional) many-valued logic as a formalisation of the logic of vague propositions. See e.g. Williamson's treatise on this point.

Artificial precision

[*Fuzzy logic*] *imposes artificial precision* [... *While*] *one is not obliged to require that a predicate either definitely applies or definitely does not apply, one is obliged to require that a predicate applies to such-and-such, rather than to such-and-such other, degree* (*e.g. that a man 5ft 10in tall belongs to* tall *to degree* 0.6 *rather than 0.5*).

S. Haack, 1979

Artificial precision

The degree theorist's assignments impose precision in a form that is just as unacceptable as a classical true/false assignment. [...] *All predications of "is red" will receive a unique, exact value, but it seems inappropriate to associate our vague predicate "red" with any particular exact function from objects to degrees of truth. For a start, what could determine which is the correct function, settling that my coat is red to degree 0.322 rather than 0.321?*

R. Keefe, 2000

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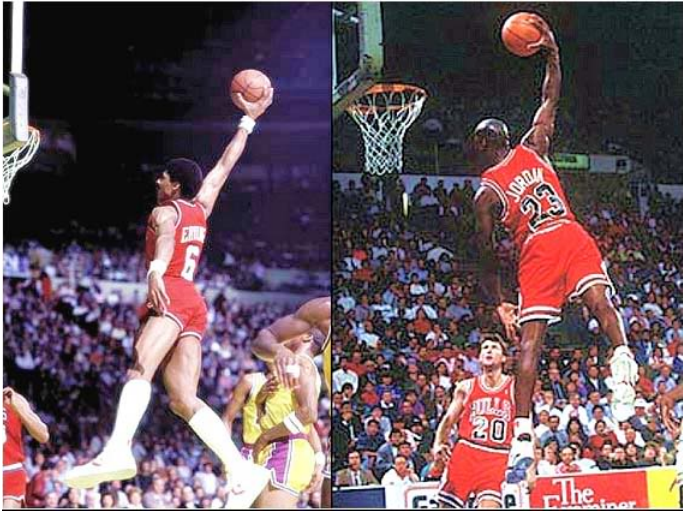
Example: The truth value of $p :=$"Enzo is tall" is the height of Enzo (=190cm) linearly renormalised over $[0, 1]$.

Taking e.g. as maximum height 250cm, and as minimum height 90cm, the truth value of p is $\frac{190-90}{250-90} = \frac{100}{160} = 0.625$.

Rebuttal:



Julius Erving Michael Jordan

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is untenable. Consider the predicate Tall, written $T(x)$. Then:

$$\text{It is the case that } T(\text{Jordan}). \qquad (\star)$$

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$$\text{Erving is taller than Jordan.} \qquad (\dagger)$$

By $(\dagger)$ and our assumptions on the renormalisation map H,

$$H(\text{Jordan}) = 1 < 1 = H(\text{Erving}), \quad \text{contradiction.}$$

There is one **counter response** to this rebuttal that is relatively common, which is however mistaken:

- X :="Jordan is tall".
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- There is now no problem with $w(X) < w(Y)$, as X and Y are distinct propositional variables. Nothing forces $w(X) = w(Y)$.

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But the point of predicate logic is precisely that predicates can be applied to a variety of terms: fixing the context etc. there should be one predicate $T(x)$ for "x is tall"—lest there be no logic at all. The counter response is worse than the original problem: it leads us to reject the possibility that there is a logic of such a monadic predicate as Tall.

The basics of Łukasiewicz logic

Jan Łukasiewicz, 1878–1956.

We start with a (finite or infinite) set of propositional variables, or atomic formulæ, that are to stand for propositions. Say, if we content ourselves with countably many:

$$X_1, X_2, \ldots, X_n, \ldots \ .$$

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To construct compound formulæ we use the logical connectives:

- $\vee$, for disjunction ("inclusive or", Latin *vel*);
- $\wedge$, for conjunction ("and", Latin *et*);
- $\rightarrow$, for implication ("if…then…", conditional assertions);
- $\neg$, for negation ("not", negative assertions).

The usual recursive definition of general formulæ now reads as follows.

- $\top$ and $\bot$ are formulæ.
- All propositional variables are formulæ.
- If α and β are formulæ, so are $(\alpha \vee \beta)$, $(\alpha \wedge \beta)$, $(\alpha \to \beta)$, and $\neg\alpha$.
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Let us write FORM for the set of all formulæ constructed over the countable language $X_1, \ldots, X_n, \ldots$. Observe that formulæ are defined exactly in the same manner in classical logic.

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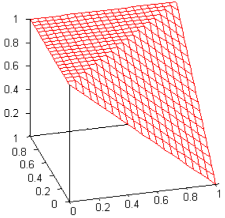

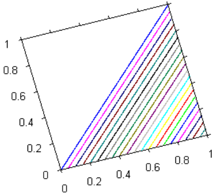

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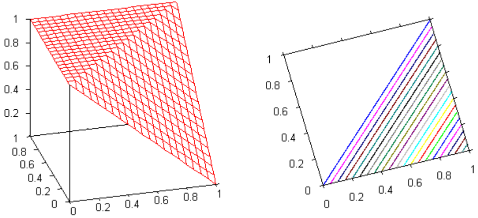

Truth-function of Łukasiewicz implication.

$$w(\alpha \to \beta) = \min\{1, 1 - (w(\alpha) - w(\beta))\}$$

We are using $\{\bot, \neg, \to\}$ only as primitive connectives. The remaining ones ($\top$, $\vee$, and $\wedge$) are definable as in classical logic. And it is customary to define more.

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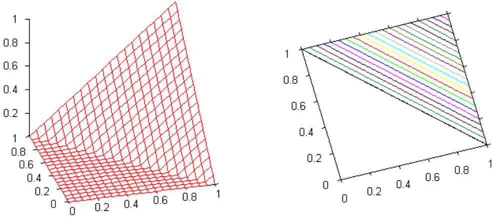
Notation	**Definition**	**Name**
$\bot$	–	*Falsum*
$\top$	$\neg\bot$	*Verum*
$\neg\alpha$	–	Negation
$\alpha \to \beta$	–	Implication
$\alpha \vee \beta$	$(\alpha \to \beta) \to \beta$	(Lattice) Disjunction
$\alpha \wedge \beta$	$\neg(\neg\alpha \vee \neg\beta)$	(Lattice) Conjunction
$\alpha \leftrightarrow \beta$	$(\alpha \to \beta) \wedge (\beta \to \alpha)$	Biconditional
$\alpha \oplus \beta$	$\neg\alpha \to \beta$	Strong disjunction
$\alpha \odot \beta$	$\neg(\alpha \to \neg\beta)$	Strong conjunction
$\alpha \ominus \beta$	$\neg(\alpha \to \beta)$	Co-implication

Table: Connectives in Łukasiewicz logic.

The corresponding formal semantics is as follows:

Notation	Formal semantics
$\bot$	$w(\bot) = 0$
$\top$	$w(\top) = 1$
$\neg\alpha$	$w(\neg\alpha) = 1 - w(\alpha)$
$\alpha \to \beta$	$w(\alpha \to \beta) = \min\{1, 1 - (w(\alpha) - w(\beta))\}$
$\alpha \vee \beta$	$w(\alpha \vee \beta) = \max\{w(\alpha), w(\beta)\}$
$\alpha \wedge \beta$	$w(\alpha \wedge \beta) = \min\{w(\alpha), w(\beta)\}$
$\alpha \leftrightarrow \beta$	$w(\alpha \leftrightarrow \beta) = 1 - \|w(\alpha) - w(\beta)\|$
$\alpha \oplus \beta$	$w(\alpha \oplus \beta) = \min\{1, w(\alpha) + w(\beta)\}$
$\alpha \odot \beta$	$w(\alpha \odot \beta) = \max\{0, w(\alpha) + w(\beta) - 1\}$
$\alpha \ominus \beta$	$w(\alpha \ominus \beta) = \max\{0, w(\alpha) - w(\beta)\}$

Table: Formal semantics of connectives in Łukasiewicz logic.



Truth-function of Łukasiewicz "strong conjunction" ⊙.
(*Note: Non-idempotent operation.*)

$$w(\alpha \odot \beta) = \max\{0, w(\alpha) + w(\beta) - 1\}$$

Analytic truths, or tautologies after L. Wittgenstein, are now defined as those formulæ $\alpha \in \textsc{Form}$ that are true in every possible world, i.e. such that $w(\alpha) = 1$ for any assignment w.

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- $\bot \to \alpha$ (*Ex falso quodlibet*)
- ~~$\alpha \vee \neg\alpha$~~ (*Tertium non datur*)
- ~~$\neg(\alpha \wedge \neg\alpha)$~~ (Principle of non-contradiction)
- $\neg\neg\alpha \to \alpha$ (Law of double negation)
- ~~$(\neg\alpha \to \alpha) \to \alpha$~~ (*Consequentia mirabilis*)
- $(\alpha \to \beta) \to (\neg\beta \to \neg\alpha)$ (Contraposition)
- $(\alpha \to \beta) \vee (\beta \to \alpha)$ (Pre-linearity)

Define: $\textsc{Taut} \subseteq \textsc{Form}$ is the set of all tautologies. Write: $\models \alpha$ to mean $\alpha \in \textsc{Taut}$.

Tautologies are a formal semantic notion. Logic is concerned with the relationship between syntax (the language) and semantics (the world).

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The syntactic counterpart of a tautology is a provable formula, also called theorem of the logic.

To define provability, we select (with a lot of hindsight) a set of tautologies, and declare that they are axioms: they count as provable formulæ by definition.

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To define provability, we select (with a lot of hindsight) a set of tautologies, and declare that they are axioms: they count as provable formulæ by definition.

Next we select a set of deduction rules that tell us that if we already established that formulæ $\alpha_1, \ldots, \alpha_n$ are provable, and these have a certain shape, then a specific formula β is also a provable formula.

Most important deduction rule (only one we use): modus ponens.

$$\frac{\alpha \qquad \alpha \to \beta}{\beta} \quad (\textsc{mp})$$

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Now we declare that a formula $\alpha \in \textsc{Form}$ is provable if there exists a proof of α, that is, a finite sequence of formulæ $\alpha_1, \ldots, \alpha_l$ a such that:

- $\alpha_l = \alpha$.
- Each α_i, $i < l$ is either an axiom, or is obtainable from α_j and α_k, $j, k < i$, via an application of *modus ponens*.

Define: $\textsc{Thm} \subseteq \textsc{Form}$ is the set of provable formulæ. Write: $\vdash \alpha$ to mean $\alpha \in \textsc{Thm}$.

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Define: $\textsc{Thm} \subseteq \textsc{Form}$ is the set of provable formulæ. Write: $\vdash \alpha$ to mean $\alpha \in \textsc{Thm}$.

We still need to define the axioms.

Axiom system for classical logic.

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(A0) $\bot \to \alpha$ *Ex falso quodlibet.*

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(A4) $(\alpha \to \beta) \to (\neg\beta \to \neg\alpha)$ Contraposition.

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(A5) $(\neg\alpha \to \alpha) \to \alpha$ *Consequentia Mirabilis.*

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Upon defining

$$\alpha \vee \beta \equiv (\alpha \to \beta) \to \beta$$

(A0–A5) read as shown next.

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(A5) $\alpha \vee \neg\alpha$ *Tertium non datur.*

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Deduction rule for classical logic.

(R1) $\dfrac{\alpha \qquad \alpha \to \beta}{\beta}$ *Modus ponens.*

Axiom system for Łukasiewicz logic.

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Deduction rule for Łukasiewicz logic.

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Such "succint descriptions" can be polysemous to a surprising extent indeed.

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Such "succint descriptions" can be polysemous to a surprising extent indeed.

Moral: The import of removing one axiom from an axiom system depends on the axiom system itself. In particular, Hilbert-style systems are of little use to analyse the structural properties of logics in terms of a specific axiomatisation.

(For this, the Gentzen-style systems used in proof theory are more useful.)

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This concludes our definition of Łukasiewicz (propositional) logic.

A first important result. In Łukasiewicz logic, the relationship between tautologies and theorems is entirely analogous to the one that holds in classical logic. It is stated in the next result, a substantial piece of mathematics:

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A first important result. In Łukasiewicz logic, the relationship between tautologies and theorems is entirely analogous to the one that holds in classical logic. It is stated in the next result, a substantial piece of mathematics:

Soundness and Completeness Theorem for Ł

$$\textsc{Taut} = \textsc{Thm}\,.$$

A. Rose and J. Barkley Rosser, *Trans. of the AMS*, 1958.

Proof is syntactic. Algebraic proof given shortly thereafter by C.C. Chang, which introduced MV-algebras for this purpose. We will return to them if time allows.

Classical logic satisfies a stronger completeness theorem. For $S, \{\alpha\} \subseteq \textsc{Form}$, write $S \vdash \alpha$ if α is provable form the logical axioms augmented by S, and $S \vDash \alpha$ if α holds in each model (=possible world, assignment) wherein each formula of S holds.

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Strong Completeness Theorem for CL

For any $\alpha \in \textsc{Form}$, and any set $S \subseteq \textsc{Form}$,

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In the actual use of any logic, it is of great importance to have completeness under additional sets S of assumptions. It is S that encodes our knowledge about a specific application domain. Pure logic ($S = \emptyset$) can teach us nothing about the world, by definition.

Łukasiewicz logic fails strong completeness.

Let S be the set of formulæ in one variable p:

$$\varphi_n(p) := ((n+1)(p^n \wedge \neg p)) \oplus p^{n+1},$$

for each integer $n \geqslant 1$, where

$$p^k := \underbrace{p \odot \cdots \odot p}_{k \text{ times}},$$

$$kp := \underbrace{p \oplus \cdots \oplus p}_{k \text{ times}}.$$

Then $S \nvdash_{Ł} p$, but $S \vDash_{Ł} p$.

$S \not\vdash_{Ł} p$, but $S \vDash_{Ł} p$.

Intuitively, you can think of S as embodying the following infinite set of assumptions:

1. p :=“Enzo is tall” is true to degree $\geqslant 1/2$.
2. p :=“Enzo is tall” is true to degree $\geqslant 2/3$.
3. p :=“Enzo is tall” is true to degree $\geqslant 3/4$.
4. ...

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- Semantically, the only possible world compatible with all of S is the one such that $w(p) = 1$, i.e. $S \vDash_{Ł} p$.

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Taking stock. $\vdash_{Ł}$ is **compact**, but $\vDash_{Ł}$ is not.

Note. $S \vdash_{Ł} \alpha \Rightarrow S \vDash_{Ł} \alpha$ always.

The Hay-Wójcicki Theorem:

Completeness Theorem for f.a. theories in Ł

For any $\alpha \in \textsc{Form}$, and any finite set $F \subseteq \textsc{Form}$,

$$F \vDash_{Ł} \alpha \quad \text{if, and only if,} \quad F \vdash_{Ł} \alpha\,.$$

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A folklore theorem:

Completeness Theorem for maximal theories in Ł

For any $\alpha \in \textsc{Form}$, and any maximal consistent set $M \subseteq \textsc{Form}$,

$$M \vDash_{Ł} \alpha \quad \text{if, and only if,} \quad M \vdash_{Ł} \alpha\,.$$

Satisfiability and consistency in Ł

Notion	Definition	Description
α is satisfiable	$\exists w$ such that $w(\alpha) = 1$	α is 1-satisfiabile
α is consistent	$\exists \beta$ such that $\alpha \nvdash_{Ł} \beta$	α does not prove smthg.
α is unsatisfiable	$\forall w$ we have $w(\alpha) < 1$	α is not 1-satisfiable
α is inconsistent	$\forall \beta$ we have $\alpha \vdash_{Ł} \beta$	α proves everything
α is strongly unsat.	$\forall w$ we have $w(\alpha) = 0$	α is always false
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Nota Bene. The terminology “Strongly unsatisfiable/inconsistent” is not standard. I only use it for ease of exposition. I do not know of a standard terminology for these concepts.

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Equivalent in classical logic by the Deduction Theorem.

Deduction Theorem for CL

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Local Deduction Theorem for Ł

For any $\alpha, \beta \in \textsc{Form}$,

$$\alpha \vdash_{Ł} \beta \quad \text{if, and only if,} \quad \exists n \geqslant 1 \text{ such that } \vdash_{Ł} \alpha^n \to \beta\,.$$

(Notation: $\alpha^n := \underbrace{\alpha \odot \cdots \odot \alpha}_{n \text{ times}}$.)

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2. The (Fregean) distinction between asserting a proposition and contemplating that proposition becomes essential: in classical logic, deduction theorem+bivalence make that distinction far less important. (Cf. the Tarskian identification of the meaning of a proposition α with its truth conditions: this fails badly in Łukasiewicz logic.)

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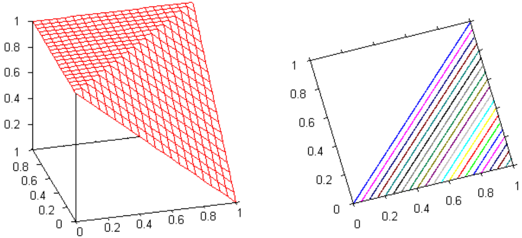
3. As a consequence of the two previous items, we will soon see that it is easy to say what the assertion $\vdash \alpha \to \beta$ means, while it is far harder to say what the plain proposition $\alpha \to \beta$ means. In other words, the intended meaning of the connective $\to$ is unclear.

Symbol	Name	Classically read
$\top$	*verum*	Always true
$\bot$	*falsum*	Always false
$\vee$	disjunction	Inclusive or (*vel*)
$\wedge$	conjunction	And
$\to$	implication	If. . . then. . .
$\neg$	negation	Not

Notation	Definition	Formal Semantics
$\top$	$\neg\bot$	$w(\top) = 1$
$\alpha \vee \beta$	$(\alpha \to \beta) \to \beta$	$w(\alpha \vee \beta) = \max\{w(\alpha), w(\beta)\}$
$\alpha \wedge \beta$	$\neg(\neg\alpha \vee \neg\beta)$	$w(\alpha \wedge \beta) = \min\{w(\alpha), w(\beta)\}$
$\alpha \leftrightarrow \beta$	$(\alpha \to \beta) \wedge (\beta \to \alpha)$	$w(\alpha \leftrightarrow \beta) = 1 - \lvert w(\alpha) - w(\beta)\rvert$
$\alpha \oplus \beta$	$\neg\alpha \to \beta$	$w(\alpha \oplus \beta) = \min\{w(\alpha) + w(\beta), 1\}$
$\alpha \ominus \beta$	$\neg(\alpha \to \beta)$	$w(\alpha \ominus \beta) = \max\{w(\alpha) - w(\beta), 0\}$

Table: Connectives in Łukasiewicz logic.



Truth-function of Łukasiewicz implication.

$$w(\alpha \to \beta) = \min\{1, 1 - (w(\alpha) - w(\beta))\}$$

$$w(\alpha \to \beta) = \begin{cases} 1 & \text{if } w(\alpha) \leqslant w(\beta) \\ 1 - (w(\alpha) - w(\beta)) & \text{otherwise.} \end{cases}$$

What does this function (intuitively) mean?

I.e., what is the meaning of implication in Łukasiewicz logic?

The Ways of Vagueness

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- Tall is vague, and it has a natural contrary, namely, Short. In symbols,

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  Similarly: Young, Beautiful, etc.

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Tall and Red are fundamentally different vague predicates.

- Tall is vague, and it has a natural contrary, namely, Short. In symbols,
$$\neg T(x) \equiv (\neg T)(x) \equiv S(x).$$
Similarly: Young, Beautiful, etc.
- Red is vague, but it does not have a natural contrary. There is no name for non-Red in the colour spectrum. Similarly: Cute, Nice, etc. I'll write $\sim R(x)$ for the negation of such predicates.

Take careful note: $\neg R(x)$ just doesn't make sense; $\sim T(x)$ does.

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- The extension of $R(x)$ is the set of entities which are a clear, indisputable case of redness.
- The extension of $\sim R(x)$ is the set of entities which are a clear, indisputable case of non-redness.
- Hence, the extension of $\sim\sim R(x)$ is the set of entities which do not qualify as a clear case of non-redness; but in general they will not qualify as a clear case of redness, either. In fact: $\sim$ fails the Double Negation Law.

The logic of Red is not compatible with Łukasiewicz logic.

- $\nvdash R(x) \vee {\sim}R(x)$

It is not the case that each x is either a clear case of redness, or else clearly a case non-redness.

Indeed, there are borderline red objects.

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Indeed, there are borderline red objects.

- $\vdash {\sim}R(x) \vee {\sim}{\sim}R(x)$

Each x is either clearly case of non-redness, or else x is red to some (non-negligible) degree.

Indeed, the set of all object is partitioned into the non-red ones, and its set-theoretic complement, i.e., the set of objects which are red to some degree.

The logic of Red is not compatible with Łukasiewicz logic.
What about Tall?

Observe that in the case of $T(x)$, too, the set of all individuals is partitioned into the set of non-tall ones, and its set-theoretic complement, i.e., the set of individuals which are tall to some degree. But we cannot express that with $\neg$ and $\vee$ alone. (We could introduce $\sim T$, but I won't discuss that here.)

- $\nvdash T(x) \vee \neg T(x)$
- $\nvdash T(x) \vee S(x)$

It is not the case that each x is either clearly tall or clearly short.

Indeed, there are average-height individuals. But unlike the situation for $R(x)$:

Observe that in the case of $T(x)$, too, the set of all individuals is partitioned into the set of non-tall ones, and its set-theoretic complement, i.e., the set of individuals which are tall to some degree. But we cannot express that with $\neg$ and $\vee$ alone. (We could introduce $\sim T$, but I won't discuss that here.)

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Moreover, Tall and Short satisfy *tertium non datur* in another sense:

> *For each individual x, either x is tall to some (non-negligible) degree, or x is short to some (non-negligible) degree.*

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We need a new connective to formalise that, i.e.:

- $\vdash T(x) \oplus S(x)$.
- $\vdash T(x) \oplus \neg T(x)$.

Moreover, Tall and Short satisfy *tertium non datur* in another sense:

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- $\vdash T(x) \oplus \neg T(x)$.

These and the previous formulæ indicate that the logic of Tall and Short is compatible with Łukasiewicz logic. Indeed, I claim that the logic of Tall and Short is Łukasiewicz logic, and no other.

Before proceeding, we need a second distinction which we already mention in connection with the deduction theorem: that between assertions and propositions.

In any logic, one thing is to consider the proposition α, and another thing is to assert the proposition α. Frege invented the Frege's assertion sign to denote the latter circumstance:

$$\vdash \alpha \quad \text{means: } \textit{It is the case that } \alpha.$$

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> *A classical proposition α is assertible under no circumstances if, and only if, $\vdash \neg\alpha$, i.e., if and only if α is a contradiction.*

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- $\nvdash T(x) \wedge \neg T(x)$
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Under no circumstances can the same individual x be a clear case of tallness and a clear case of shortness.

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Indeed, if you insist on $\vdash \neg(T(x) \wedge \neg T(x))$ then you must be ready to declare each individual x either a clear case of tallness, or a clear case of shortness.

Taking stock:

> *Even though* Tall *is a vague predicate, whenever you assert* $T(x)$, *i.e.*
>
> $$\vdash T(x)$$
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> *you are committing yourself to the fact that you regard* x *as a clear, indisputable case of tallness.*

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There is no hedging when it comes to asserting vague propositions.

Question. If $T(x)$ is as before, and $Y(x)$ is the vague predicate Young, how do you read the proposition

$$T(x) \to Y(x)$$

in Łukasiewicz logic?

The following is easy to read:

$$\vdash T(x) \to Y(x).$$

It just means: *x is less a case of tallness than x is a case of youth.* However,

$$T(x) \to Y(x)$$

itself does not mean the latter. Indeed, the latter is a classical proposition, whereas $T(x) \to Y(x)$ ought to be a vague proposition.

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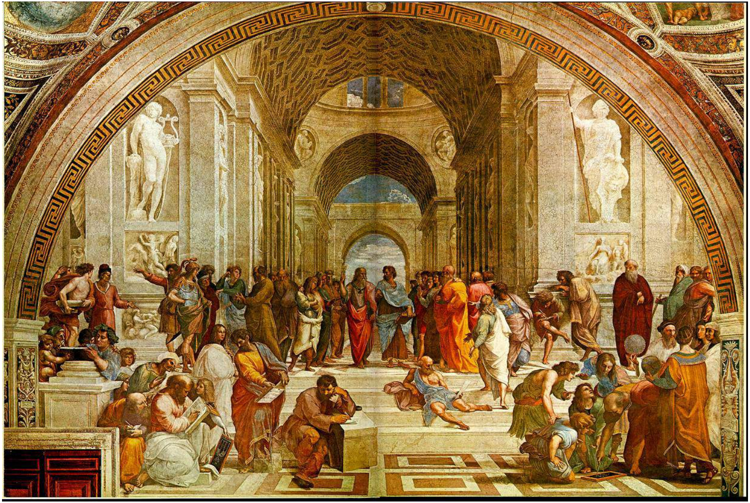
In general, while the assertion $\vdash \alpha \to \beta$ does mean α *is less true than* β in any many-valued (t-norm based) logic, the proposition $\alpha \to \beta$ does not. To argue that a natural semantics for Łukasiewicz logic is that of vague predicates with natural contraries, such as Tall, the real challenge is to explain away the monoidal connectives $\odot$ and $\oplus$, or, equivalently, their adjoint implications $\to$ and $\ominus$.

True, Truer, Much Truer



Raffaello Sanzio, La Scuola di Atene, *ca. 1509.*

E. Casari, *Comparative logic*, Synthese, 1987.

ETTORE CASARI

COMPARATIVE LOGICS

1. INTRODUCTION

Comparative Logic was created by Aristotle at the very beginnings of logic. In the *Topics* he developed, in particular, a highly satisfactory theory of the nine kinds of propositions which arise by crossing comparisons of majority, minority and equality (**μᾶλλον, ἧττον, ὁμοίως**) with situations in which 'one is said of two (**ἑνὸς περὶ δύο λεγομένου**)', 'two are said of one (**δυοῖν περὶ ἑνὸς λεγομένων**)' and 'two are said of two (**δυοῖν περὶ δύο λεγομένων**)', i.e.,

x is more A than y;
x is less A than y;
x is as much A as y;
x is more A than B;
x is less A than B;
x is as much A as B;
x is more A than y is B;
x is less A than y is B;

Aristotle's example, in the *Topics*, of inference with *comparatives of comparatives*.

P1. x is more T than z.

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P3. x is more (more T than z) than (that by which y is more T than z).

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A variation of Aristotle's example aimed at a degree-based treatment of vagueness:

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Easy Exercise: Work out the meaning of $\neg((\alpha \ominus \beta) \wedge (\beta \ominus \alpha))$.

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Easy Exercise: Work out the meaning of $\neg((\alpha \ominus \beta) \wedge (\beta \ominus \alpha))$.

Tricky Exercise: Work out the meaning of $\alpha \to \beta$, $\alpha \oplus \beta$, and $\alpha \odot \beta$ using functional completeness of $(\neg, \ominus, \bot)$. E.g., $\neg(\alpha \ominus \beta) \equiv \alpha \to \beta$. Does the result warrant commutativity of $\oplus$ and $\odot$? Does the result warrant calling $\odot$ a conjunction, or $\oplus$ a disjunction?

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$$\frac{\gamma \to \alpha \qquad \gamma \to \beta \qquad (\beta \ominus \gamma) \to (\alpha \ominus \gamma)}{\beta \to \alpha} \qquad (*)$$

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Semantically, over $[0,1] \subseteq \mathbb{R}$, if $w(\alpha) = r$, $w(\beta) = s$, and $w(\gamma) = t$:

$$\frac{t \leqslant r \qquad t \leqslant s \qquad s - t \leqslant r - t}{s \leqslant r}$$

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The inference rule $(*)$ is indeed sound in Łukasiewicz logic

Summing Up

I claim that the logic of vague (monadic) predicates T that:

1. Can be universally compared as in: $T_1(x)$ is more true than $T_2(x)$, and $T_1(x)$ is much more true than $T_2(x)$, and
2. Have unique natural contraries $S \equiv \neg T$ such that $T(x)$ is more true than $T(y)$ iff $S(y)$ is more true than $S(x)$,

is Łukasiewicz infinite-valued propositional logic. A full argument will of course require more work than could be illustrated in this talk.

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But is this not just philosophical speculation? Why is this important at all for mathematical logicians and computer scientists who might want to apply Łukasiewicz logic, anyway?




WEAR APPROVED EYE PROTECTION
Made In USA


WESTCOTT
STAINLESS STEEL
USA

MV-algebras

C. C. Chang in Rome, 1969.

MV-algebras

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Lindenbaum's Equivalence Relation

Say $\alpha, \beta \in \textsc{Form}$ are logically equivalent if $\vdash \alpha \leftrightarrow \beta$. Write $\alpha \equiv \beta$.

On the quotient set $\frac{\textsc{Form}}{\equiv}$, the connectives induce operations:

- $0 := [\bot]_{\equiv}$
- $\neg[\alpha]_{\equiv} := [\neg\alpha]_{\equiv}$
- $[\alpha]_{\equiv} \oplus [\beta]_{\equiv} := [\alpha \oplus \beta]_{\equiv}$

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> 'MV-algebra' is short for 'Many-Valued Algebra', *"for lack of a better name."*
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MV-algebras : Łukasiewicz logic = Boolean algebras : Classical logic

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MV-algebras : Łukasiewicz logic = Boolean algebras : Classical logic

Abstractly: $(M, \oplus, \neg, 0)$ is an MV-algebra if $(M, \oplus, 0)$ is a commutative monoid, $\neg\neg x = x$, $1 := \neg 0$ is absorbing for $\oplus$ ($x \oplus 1 = 1$), and, characteristically,

$$\neg(\neg x \oplus y) \oplus y = \neg(\neg y \oplus x) \oplus x \qquad (*)$$

Any MV-algebra has an underlying distributive lattice bounded below by 0 and above by 1. Joins are given by

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$$x \wedge y := \neg(\neg x \vee \neg y)$$

Boolean algebras=Idempotent MV-algebras: $x \oplus x = x$.
Equivalently: MV-algebras that satisfy the *tertium non datur* law

$$x \vee \neg x = 1$$

.

The interval $[0, 1] \subseteq \mathbb{R}$ can be made into an MV-algebra with neutral element 0 by defining

$$x \oplus y := \min\{x + y, 1\} \ , \quad \neg x := 1 - x \, .$$

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This means: The class of MV-algebras coincides with $\mathrm{HSP}([0,1])$ — any MV-algebra can be represented as a homomorphic image of a subalgebra of a product of copies of $[0,1]$.
Or: The equations (in the language of MV-algebras) that hold in all MV-algebras are exactly those that hold in $[0,1]$.

Or: Any $\alpha \in$ Form that has a counter-model in some MV-algebra, already has a counter-model in $[0,1]$.

Let us consider the *tertium non datur* equation:

$$x \vee \neg x = 1 . \qquad (\star)$$

Then $(\star)$ is not an identity over $[0,1]$: the only evaluations into $[0,1]$ that satisfy $(\star)$ are $x \mapsto 0$ and $x \mapsto 1$ — the Boolean, or classical, evaluations.

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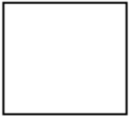
The boundary of the unit square.

$$X \vee \neg X = 1 \qquad (\star)$$

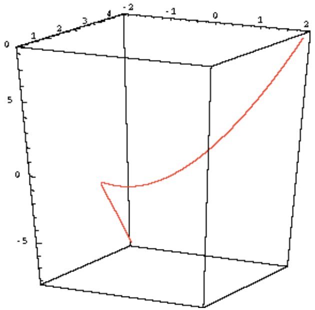
The boundary of the unit interval.

$$X \vee \neg X \vee Y \vee \neg Y = 1 \qquad (\star\star)$$



The boundary of the unit square.

The *twisted cubic*: $\mathbb{V}(\{y - x^2, z - x^3\})$



(Parametrisation: $t \longmapsto (t, t^2, t^3)$.)

Rational polyhedra

Leonardo's Truncated Icosahedron

(*Illustration for Luca Pacioli's The Divine Proportion, 1509.*)

We consider finitely presented MV-algebras, *i.e.* those of the form $\mathscr{F}_n/\theta$, with θ a finitely generated congruence (ideal). The assumption on θ is far from immaterial: there is no Hilbert's Basis Theorem for MV-algebras.

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The convex hull of a set $P \subseteq \mathbb{R}^n$, written conv P, is the collection of all convex combinations of elements of P:

$$\text{conv } P = \left\{ \sum_{i=1}^{m} r_i v_i \mid v_i \in P \text{ and } 0 \leqslant r_i \in \mathbb{R} \text{ with } \sum_{i=1}^{m} r_i = 1 \right\}.$$

Such a set is convex if $P = \text{conv } P$.

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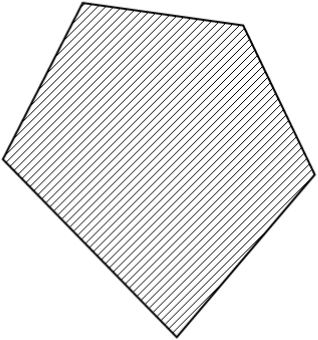
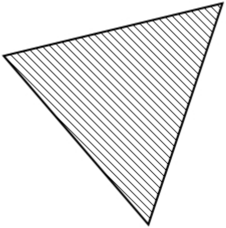
The set P is called:

- a polytope, if there is a finite $F \subseteq \mathbb{R}^n$ with $P = \operatorname{conv} F$;
- a rational polytope, if there is a finite $F \subseteq \mathbb{Q}^n$ with $P = \operatorname{conv} F$.

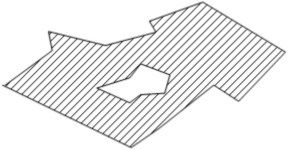


A polytope in $\mathbb{R}^2$.



A polytope in $\mathbb{R}^2$ (*a* simplex).

A (compact) polyhedron in $\mathbb{R}^n$ is a union of finitely many polytopes in $\mathbb{R}^n$.

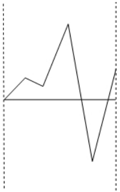


A polyhedron in $\mathbb{R}^2$*.*

Similarly, a rational polyhedron is a union of finitely many rational polytopes.

Let $P \subseteq \mathbb{R}^n$ be a rational polyhedron. A continuous function $f \colon P \to \mathbb{R}$ is a $\mathbb{Z}$-map if the following hold.

1. There is a finite set $\{L_1, \ldots, L_m\}$ of affine linear functions $L_i \colon \mathbb{R}^n \to \mathbb{R}$ such that $f(x) = L_{i_x}(x)$ for some $1 \leqslant i_x \leqslant m$.



A piecewise linear function $[0, 1] \to \mathbb{R}$.

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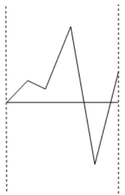

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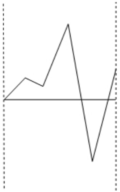

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A map $F: P \subseteq \mathbb{R}^n \to Q \subseteq \mathbb{R}^m$ between polyhedra always is of the form $F = (f_1, \ldots, f_m)$, $f_i: P \to \mathbb{R}$. Then F is a $\mathbb{Z}$-map if each one of its scalar components f_i is.

Rational polyhedra are precisely the subsets of $\mathbb{R}^n$ that are definable by a term in the language of MV-algebras; and $\mathbb{Z}$-maps are precisely the continuous transformations that are definable by tuples of terms in that language.

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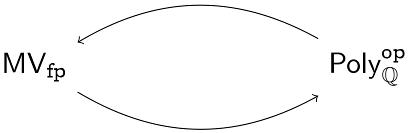
Stone-type duality for finitely presented MV-algebras

The category of finitely presented MV-algebras, and their homomorphisms, is equivalent to the opposite of the category of rational polyhedra, and the $\mathbb{Z}$-maps amongst them.

- V.M. & L. Spada, *Duality, projectivity, and unification in Łukasiewicz logic and MV-algebras*, Annals of Pure and Applied Logic, 2012.



From MV-algebras to rational polyhedra: Given $\mathscr{F}_n/\langle\tau(x_1,\ldots,x_n)\rangle$, the associated rational polyhedron $\mathbb{V}(\tau)$ is the set of n-tuples $(r_1,\ldots,r_n)\in[0,1]^n$ such that $\tau(r_1,\ldots,r_n)=0$ in $[0,1]$.

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Example. If $\tau(x_1,\ldots,x_n)$ is identically equal to 0 in any MV-algebra, then it generates the trivial ideal $\{0\}$. In this case, $\mathscr{F}_n/\langle\tau\rangle=\mathscr{F}_n$, and $\mathbb{V}(\tau)=[0,1]^n$. Hence the duals of free algebras are the unit cubes.

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Remark. The subspace $\mathbb{V}(\tau)\subseteq[0,1]^n$ homeomorphic to the maximal spectral space of $\mathscr{F}_n/\langle\tau\rangle$, topologised by the (analogue of) the Zariski topology. The MV-algebra $\nabla(P)$ is the exact analogue for rational polyhedra of the coordinate ring of an affine algebraic variety.

$$X \vee \neg X \vee Y \vee \neg Y = 1$$

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For more on MV-algebras and polyhedral geometry, the measure theory and the geometry of polyhedral fans in lattice-groups, and sheaf-theoretic representations, see e.g.:

1. M. Gehrke, S. van Gool, and V.M., *Sheaf representations of MV-algebras and lattice-ordered abelian groups via duality*, submitted to the Israel J. Math., 2013.
2. V.M., *Lattice-ordered Abelian groups and Schauder bases of unimodular fans, II*, Trans. of the AMS, 2013.
3. C. Manara, V.M., and D. Mundici, *Lattice-ordered Abelian groups and Schauder bases of unimodular fans*, Trans. of the AMS, 2007.
4. V.M., *The Lebesgue state of a unital abelian lattice-ordered group, II*, J. Group Theory, 2009.
5. V.M., D. Mundici, *The Lebesgue state of a unital abelian lattice-ordered group*, J. Group Theory, 2007.

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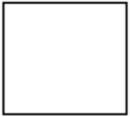
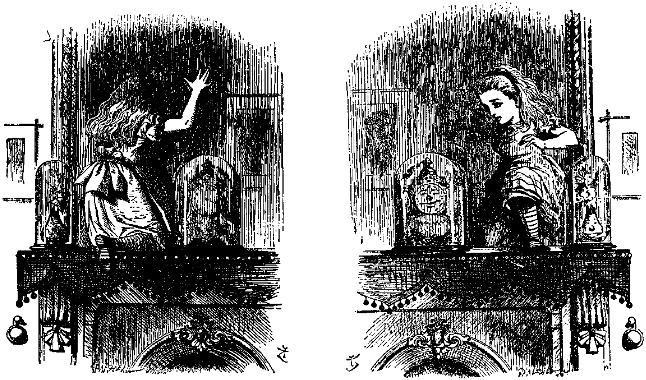


Stone-type duality for finitely presented MV-algebras

$$\mathsf{MV}_{\mathrm{fp}} \rightleftarrows \mathsf{Poly}^{\mathrm{op}}_{\mathbb{Q}}$$

Duality



$$\mathsf{BoolAlg} \rightleftarrows \mathsf{StoneSp}^{\mathrm{op}}$$

In the thirties, Stone discovered that the set of maximal ideals of a Boolean algebra carries a natural topology: open sets correspond to arbitrary ideals. In the Introduction to his book on Stone spaces, P. Johnstone writes:

> *Now this was a really bold idea. Although the practitioners of abstract general topology [...] had by the early thirties developed considerable expertise in the construction of spaces with particular properties, the motivation of the subject was still geometrical [...] and (as far as I know) nobody had previously had the idea of applying these techniques to the study of spaces constructed from purely algebraic data.*

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The ensuing spaces are nowadays called Stone spaces. The clopen sets — those sets which are both closed and open in the topology — correspond to principal ideals, and hence to elements of the algebra. Thus, the original algebra can be recovered from its space of maximal ideals; Stone's construction is in fact a two-way road.

The syntax-semantics dictionary.

Algebra, or Syntax.	Topology, or Semantics.
Boolean algebra	Stone (or Boolean) space
Homomorphism	Continuous map
Finite Boolean algebra	Finite set
Finite algebra homomorphism	Function
Free n-gen. algebra	$\{0,1\}^n$
Maximal ideal	Point of Stone space
Ideal	Closed subset of Stone space
Principal ideal	Clopen subset of Stone space
$\vdots$	$\vdots$

$$\mathsf{BoolAlg} \rightleftarrows \mathsf{StoneSp}^{\mathrm{op}}$$

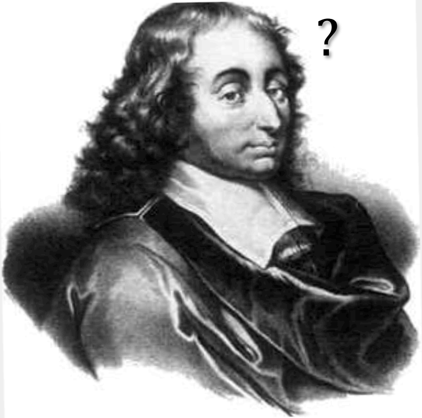
The syntax-semantics dictionary.

Algebra, or Syntax.	Geometry, or Semantics.
F.p. algebra	Rational polyhedron
Homomorphism	$\mathbb{Z}$-map
F.p. subalgebra	Continuous image by $\mathbb{Z}$-map
F.p. quotient algebra	Rational subpolyhedron
F.p. projective algebra	Retract of cube by $\mathbb{Z}$-maps
Free n-gen. algebra	$[0,1]^n$
Maximal congruence	Point of rational polyhedron
Intersection of maximal cong.	Closed subset of rational polyhedron
Finite product $A \times B$	Finite disjoint union
$\vdots$	$\vdots$

$$\mathsf{MV}_{\mathrm{fp}} \rightleftarrows \mathsf{Poly}_{\mathbb{Q}}^{\mathrm{op}}$$

Algebraic geometry	General algebra
Ground field k	A
$k[x_1, \ldots, x_n]$	$\mathscr{F}_n$
Affine space k^n	A^n
Ideal of $k[x_1, \ldots, x_n]$	Congruence on $\mathscr{F}_n$
Affine variety in k^n	Galois-fixed subset of A^n
Coord. ring $k[x_i]/\mathbb{I}(\mathbb{V}(S))$	Quotient $\mathscr{F}_n/\mathbb{I}(\mathbb{V}(S))$
Homomorphism of k-alg.	V-homomorphism
Map of affine varieties	Term-definable map
Nullstellensatz	V.M. & L. Spada, 2012
co-*Nullstellensatz*	?
Maximal ideal	Maximal congruence
⋮	⋮

- V.M. and L. Spada, *The Dual Adjunction between MV-algebras and Tychonoff spaces*, Studia Logica 100, *in memoriam* Leo Esakia, 2012.
- O. Caramello, V.M., and L. Spada, *General affine adjunctions, and Nullstellensätze*, in preparation.



Blaise Pascal (1623 – 1662)

Epilogue

Betting on Vague Propositions?

It is the eve of the 2006 World Cup Final at Berlin’s Olympic Stadium:

Italy is going to play France.

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$E =$ "Italy scores in the match against France"

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Blaise's Stake

Proposition	Stake
E	1€
not E	0€
...	...

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...	...

Ada's Book

Proposition	Price of Bet
E	0.25
not E	0.75
...	...

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Ada Lovelace (1815 – 1852)

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25 cents now. →

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Blaise Pascal (1623 – 1662)

25 cents now.

1€ if Italy scores,
0€ otherwise

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Probability Theory deals with Events described by Formulae in Classical Logic (=Boolean algebras of Events).

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E	1€
not *E*	…
…	…

Ada's Book

Proposition	Price of Bet
E	**?**
not *E*	**?**
…	…

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Proposition	Stake
E	1€
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...	...

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Proposition	Price of Bet
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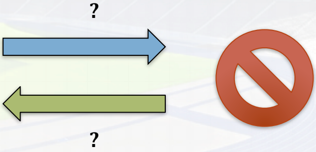

Ada Lovelace (1815 – 1852)

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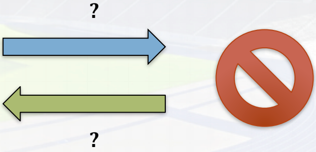
Is There a Probability Theory of Events described by Vague Propositions?

Blaise Pascal (1623 – 1662)



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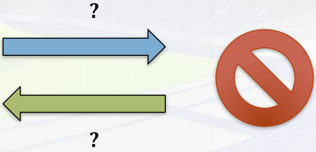
Is there a logic of Vague Propositions?
Is it just Classical Logic?
If not, which Non-Classical Logic is it?

Blaise Pascal (1623 – 1662)



Ada Lovelace (1815 – 1852)

Probability Theory of Non-Classical Events

FIRB Futuro in Ricerca 2012–2016

http://www.pnce.unimi.it

Probability Theory of Non-Classical Events

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Grazie dell'attenzione.