A fixed point theory over stratified truth

August 22, 2023

Today. . .

I look back to μ-calculus and take (loose) inspiration from 's work on intensional fixed points (see e.g. GJ's work IPA(σ) in APAL 2004 over 1st order arithmetic).

IPA(σ) allows the building up of fixed points in a very nested and entangled way and it is is classified by the Feferman-Schütte ordinal Γ_0.

Here we experiment with much stronger systems and and address the question:

to what extent is a stratified (implicitly type theoretic) discipline compatible with self-reference or unfoundedness.

The stratified framework ST_μ

In Quine's NF those instances of comprehension asserting the existence of $\{x|A\}$ are accepted which can be **stratified**. This means that there is a function σ from the variables appearing in A to the natural numbers such that in each subformula $x \in y$ one has $\sigma(x) + 1 = \sigma(y)$, and $\sigma(x) = \sigma(y)$ in each subformula $x = y$. The idea is that only those instances of comprehension are acceptable which make sense in simple type theory.
We adapt this idea to the theory of truth as well. The question is: to what extent is a stratified (implicitly type theoretic) discipline compatible with self-reference or unfoundedness.

A stratified μ-calculus ST_μ: the language

We define an extension of the theory of stratified truth ST, where we have terms representing fixed points of stratified monotone operations.

The language includes: (i) variables; an individual constant 0; (ii) logical operations $\neg$, $\wedge$, $\forall$; (iii) predicate symbols T (unary), $=$ (binary); (iv) unary function symbols *tr*, *neg*, *all*, *suc*, *left*, *right*; the binary function symbols *id*, *pred*, *and*, *pair*; in addition the binding operators μ and $[-|-]$.

NB: If $\vec{x}$ is a list (possibly empty) of variables, y in an additional variable, one has to simultaneously inductively define the notions of (i) formula; (ii) formula positive (negative) in y; (iii) term; (iv) term positive (negative) in y.

Representing formulas by terms

Define $A \mapsto [A]$ with $FV(A) = FV([A])$:

- $[t = s] := id(t, s)$;
- $[T(t)] := tr(t)$
- $[\neg A] := neg([A])$;
- $[A \wedge B] := and([A], [B])$;
- $[\forall x A] := all([x \mid A])$

Stratification of terms and formulas

If E is an expression, E is *stratified* iff it is possible to assign a natural number (a type in short) to each term occurrence and to each T-occurrence of E, so that:

- all free occurrences of the same variable in any subexpression of E have the same type;
- in each expression of the form $pred(t, s)$ the type of t is one greater than the type of its argument s; $pred(t, s)$ is assigned the type of t;
- each expression of the form $tr(t)$ is assigned a type one greater than the type of t; in each expression of the form $T(t)$, T is assigned a type one greater than the type of t;
- in each expression of the form $t = s$, $id(t, s)$, $pair(t, s)$, t has the same type as s; $id(t, s)$ is assigned the same type of t (and hence of s);
- each expression of the form $neg(t)$, $all(t)$, $suc(t)$, $left(t)$, $right(t)$ is assigned the same type as t.

- each expression of the form $and(t, s)$, $pair(t, s)$ is assigned the same type as the type of t, s (that must have received the same type);
- each term of the form $[x \mid C]$ is assigned a type one greater than the type assigned to x, and all the free occurrences of x in C receive the same type;
- in each expression of the form $\forall xA$, if x is free in A, then the free occurrences of x in A and the occurrence of x in $\forall x$ receive the same type;
- each term of the form $\mu yt(y, \vec{x})$ is assigned the same type as y and t, and all the parameters $\vec{x}$ in t receive the same type.

In general A formula (term) is $n+1$-stratified iff it is stratified by means of $0, \ldots, n$

T-axioms for ST_μ

$$
\begin{aligned}
&T(id(x,y)) \leftrightarrow x = y;\\
&T(neg(id(x,y)) \leftrightarrow \neg x = y;\\
&T(tr(x)) \leftrightarrow T(x);\\
&T(neg(tr(x)) \leftrightarrow \neg T(x);\\
&T(neg(neg(x)) \leftrightarrow T(x);\\
&T(and(x,y)) \leftrightarrow T(x) \wedge T(y);\\
&T(neg(and(x,y)) \leftrightarrow T(neg(x)) \vee T(neg(y));\\
&T(all(f)) \leftrightarrow \forall x T(pred(f,x));\\
&T(neg(all(f)) \leftrightarrow \exists x T(neg(pred(f,x)))
\end{aligned}
$$

Remark

The clauses involving $T(neg(tr(x))$ and predication are *strongly non-kripkean* and make the truth predicate closer to its classical counterpart.

Consistency, β-conversion...

- T-consistency:
$$\neg(T(a) \wedge T(neg(a)))$$
- T is well-defined on predication:
$$T(pred(f,x)) \vee T(neg(pred(f,x)))$$
- Stratified β-conversion: if A is stratified,
$$\begin{aligned} T(pred([x|A],u)) &\leftrightarrow T([A[x := u]]) \\ T(neg(pred([x|A],u))) &\leftrightarrow T([\neg A[x := u]]) \end{aligned}$$
- Pairing axioms with projections, and axioms stating that the basic constructors and the logical constructors are *injective*, *not surjective* and their images are *disjoint*; in particular $suc(x) \neq 0$.

μ-axioms for ST_μ

Let t, s be stratified, $Pos(y)$ ($=$ positive in y) and possibly depending on parameters z. Then:

- $t(\mu y.t(y,z),z) = \mu y.t(y,z)$;
- $\forall z[\forall u(t(u,z) = s(u,z)) \rightarrow \mu y.t(y,z) = \mu y.s(y,z)]$
- $\forall z \forall c[Clos_{t,z}(c) \rightarrow \mu y.t(y,z) \subseteq c]$

NB: We can add the largest fixed operation ν as well, with dual axioms, to the effect that $\nu y.t(y,z)$ is the largest fixed point of t if t is stratified and $Pos(y)$.

Immediate consequences of the axioms

Proposition

ST_μ it proves, for some closed term L:

$$\neg T(L) \wedge \neg T(neg(L))$$

Morover:

$$T(\neg T(L) \wedge \neg T(neg(L)))$$

Proof.

Since $neg(x)$ is stratified and $Pos(x)$, $L = neg(L) = \mu y.neg(y)$. Then apply logic, T-consistency and the axioms relating T with *tr*, *neg* and *and*. □

Hence T is provably internally undefined on (the simplest variant of) the Liar; and T *internally believes* this fact.

If we apply stratified β-conversion in the case of $\forall$, the following *schematic* versions of the classical conditions hold:

Lemma

If A and B are arbitrary, C is stratified, then ST_μ proves:

$$T[\neg A] \leftrightarrow \neg T[A]$$
$$T[A \wedge B] \leftrightarrow T[A] \wedge T[B]$$
$$T[\forall x C] \leftrightarrow \forall x C$$

Proposition (Uniform stratified T-schema)

If A is stratified, ST_μ *proves:*

$$\forall x(T([A(\vec{x})]) \leftrightarrow A(\vec{x}))$$

Proof: By simultaneous induction on A and the previous lemma. Moreover, ST_μ *provably believes that it is two-valued and consistent* and that each closure condition is also internally true:

(i) ST_μ proves:

$$T([T(a) \quad \vee \quad \neg T(a)]);$$
$$T([\neg(T(a) \quad \wedge \quad T(neg(a)))])$$

(ii) Moreover, if Axiom is an instance of a compositional T-axiom or T-welldefinedness, ST_μ proves $T([\text{Axiom}])$.

ST_μ with numbers

If we apply the μ axioms, we can define N and $<_N$ such that

Proposition (ST_μ)

1. $0 \in N \wedge \forall x(x \in N \rightarrow suc(x) \in N)$
2. if $A(x)$ is stratified,
   $A(0) \wedge (\forall x \in N)(A(x) \rightarrow A(suc(x)) \rightarrow \forall x \in N.A(x)$
3. $<_N$ is irreflexive, transitive and connected on N; it satisfies:
   $0 <_N a$; $a <_N suc(a)$; $a <_N suc(a) \leftrightarrow a \leq_N b$

On the problem *homogeneous vs. heterogeneous*

An essential restriction in ST_μ is that $\mu y.t(y,z)$ is homogeneously stratified, i.e. $\mu y.t(y,z)$, $t(y,z)$, y, z are assigned the same type level; this is the reason why the type raising operation $x \mapsto tr(x)$ has no fixed point. This is a strong limitation upon the self-referential abilities of the system.

According to Tupailo (LC 2005), NF is consistent relative to NF with urelemente, homogeneous pairing and the statement: there is a least fixed point of the the power set operation. Note that the power set operation if TYPE-Raising

On the strength of ST_μ: embedding the μ-calculus over arithmetic into ST_μ

Lubarsky (1993) has introduced μ-calculus over Peano arithmetic PA(μ). PA(μ) is proof-theoretically very strong (see Moellerfeld's Ph.D. thesis).

Theorem (Lower bound on ST_μ)

PA(μ) *is interpretable in* ST_μ

Proof.

Choose the set N of Fregean numbers as domain of first order variables and the subsets of N as domain of second order variables. Then apply μ-axioms in ST_μ... □

On the strength of ST_μ: upper bounds

Theorem

1. ST_μ *is interpretable in* NF.
2. *If predication is restricted to mildly impredicative formulas,* ST_μ *is interpretable into the consistent subsystem* NFI *of* NF.

Of course stratified comprehension is applied in order to interpret the predication schema; extensionality is required in order to get the pairing axioms with homogeneous stratification.
If one tries to follows the reverse path – from truth to sets –, the problem is of course how to verify a corresponding extensionality axiom for predicates.

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The set-theoretic embedding σ

$(-) \mapsto (-)^\sigma$ of terms and formulas of ST_μ into NF; below we use $(-,-)$, Q_1, Q_2, W for the corresponding set theoretic notions of definitions

$$
\begin{aligned}
(suc(t))^\sigma &= (t^\sigma) + 1 \\
(pair(t,s))^\sigma &= (t^\sigma, s^\sigma) \\
(left(t))^\sigma &= Q_1(t^\sigma) \\
(right(t))^\sigma &= Q_2(t^\sigma) \\
(pred(t,s))^\sigma &= [s^\sigma \in t^\sigma] \\
(T(t))^\sigma &= t^\sigma \in W \\
(tr(t))^\sigma &= [t^\sigma \in W] \\
(\mu y.t(y,\vec{x}))^\sigma &= \{u | \forall z (t^\sigma(z,\vec{x}) \subseteq z \to u \in z)\}
\end{aligned}
$$

Terms and formulas of ST_μ; *Pos* and *Neg*

(i) variables and the individual constant 0 are terms;

(ii) 0, every variable v occurring in $\vec{x}$ and every variable y not occurring in $\vec{x}$ are positive in the list $\vec{x}$; if $y \notin FV(E)$, E being an expression, then E is $Pos(y)$;

(iii) if t, s are terms, then
  - $t = s$, Tt are formulas;
  - $all(t)$, $suc(t)$, $neg(t)$, $tr(t)$, $left(t)$, $right(t)$ are terms, as well as $id(t, s)$, $and(t, s)$, $pair(t, s)$, $pred(t, s)$;

(iv) if A, B are formulas, then $\neg A$, $A \wedge B$, $\forall xA$ are formulas, and $FV(\forall xA]) = FV(A - \{x\})$; if A is a formula, $[x|A]$ is a term such that $FV([x|A]) = FV(A - \{x\})$;

(v) if t is positive (negative) in $\vec{x}$ and s is positive (negative) in $\vec{x}$, then $pair(t,s)$, $and(t,s)$, $id(t,s)$, $pred(t,r)$, $left(t)$, $right(s)$, $suc(t)$, $neg(t)$, $all(t)$ are all positive (negative) in $\vec{x}$;

(vi) if t is positive (negative) in $\vec{x}$, $T(pred(t,s))$ is positive (negative) in $\vec{x}$;

(vii) if t is positive (negative) in $\vec{x}$, then $T(neg(pred(t,s)))$ is negative (positive) in $\vec{x}$;

(viii) if A is $Pos(\vec{x})$ $(Neg(\vec{x}))$, then $\neg A$ is $Neg(\vec{x})$ $(Pos(\vec{x}))$; if A, B are $Pos(\vec{x})$ $(Neg(\vec{x}))$, then $A \wedge B$, $\forall vA$, $[y|A]$ are $Pos(\vec{x})$ $(Neg(\vec{x}))$ (provided y not occurring in $\vec{x}$);

(viii) if t is a term with y free and positive in y, $\mu y.t$ is a term where y is bound; moreover, if $\vec{x}$ is different from y and t is $Pos(\vec{x})$ $(Neg(\vec{x}))$, $\mu y.t(y,\vec{x})$ is $Pos(\vec{x})$ $(Neg(\vec{x}))$.

The μ-clause

- if t is a term with y free and positive in y, $\mu y.t$ is a term where y is bound; moreover, if z is different from y and t is $Pos(z)$ ($Neg(z)$), $\mu y.t(y,z)$ is $Pos(z)$ ($Neg(z)$). Informally this is because z is in Neg(z) in the clause $Clos_{t,z}(c)$, i.e. $\forall x(x \in t(c,z^+) \to x \in c)$ and hence is in $Pos(z)$ in the minimality clause:

$$Clos_{t,z}(c) \to \mu y.t(y,z) \subseteq c$$

NF-memo

- $\mathrm{NF}_k :=$ NF with comprehension for k-stratified formulas;
- A stratified term $[x \mid \varphi(x, \vec{y})]$ is *mildly impredicative* iff for some type $i \in \omega$, $[x \mid \varphi(x, \vec{y})]$ has type $i + 1$, no (free or bound) variable of $\varphi(x, \vec{y})$ is assigned type greater than $i + 1$; if every quantified variable is assigned type at most i, the term is *predicative*.
- NFI (NFP) is NF with stratified comprehension restricted to mildly impredicative conditions (predicative) conditions.

Theorem

- NF_3(*pair*)*, i.e.* NF_3 *with an axiom stating the existence of an homogeneous pairing operations is equivalent to full* NF*.*
- NFI *is consistent (in primitive recursive arithmetic plus the 1-consistency of full second order arithmetic; by Crabbé 1982, Holmes 1995).*

Numbers and Quine's pairing

Theorem (NFI)

1. N is the least set closed under $x \mapsto x + 1$ such that $0 \in N$, where

$$\begin{aligned} 0 &= \{\emptyset\} \\ a+1 &= \{x \cup \{y\} | x \in a \wedge y \notin x\} \end{aligned}$$

2. there exists a surjective homogeneously stratified paring $x^i, y^i \mapsto (x^i, y^i)^i$ with projections LEFT, RIGHT, which is $\subseteq$-monotone in each variable [see Scott et alii, BSL 2008]

Definition (Quine, JSL 1945)

$$\begin{aligned}
\phi(a) &= \{y \mid y \in a \wedge y \notin \mathcal{N}\} \cup \{y+1 | y \in a \wedge y \in \mathcal{N}\}; \\
\theta_1(a) &= \{\phi(x) \mid x \in a\}; \\
\theta_2(a) &= \{\phi(x) \cup \{0\} \mid x \in a\}; \\
(a, b) &= \theta_1(a) \cup \theta_2(b); \\
Q_1(a) &= \{z \mid \phi(z) \in a\}; \\
Q_2(a) &= \{z \mid \phi(z) \cup \{0\} \in a\}
\end{aligned}$$

NB: The given terms are positive in a, b ...

Justifying the μ-axioms

Lemma

Let $E(\vec{x})$ be a term or a formula in $Pos(\vec{x})$ ($Neg(\vec{x})$), where $\vec{x} := x_1, \ldots x_n$. Let $\vec{a} \subseteq \vec{b}$ stand for $a_1 \subseteq b_1, \ldots a_n \subseteq b_n$. If $E(\vec{x})$ is stratified and $Pos(\vec{x})$ ($Neg(\vec{x})$), then we have, provably in ST_μ:

- $\vec{a} \subseteq \vec{b} \to E(\vec{a}) \subseteq E(\vec{b})$ (resp. $\vec{b} \subseteq \vec{a} \to E(\vec{a}) \subseteq E(\vec{b})$)

The proof is by induction on the definition of E, using the properties of the homogeneous pairing operation.

ST_μ with numbers

Definition

- $pair(t, s) := (t, s)$, $left(t) := (t)_0$, $right(t) := (t)_1$
- $x \in a = T(pred(a, x))$
- $N = \mu y.[x | x = 0 \vee \exists z(z \in y \wedge x = suc(z))]$
- $<_N = \mu y.r(y)$ where

$$r(y) = [x | x = ((x)_0, (x)_1) \wedge ((x)_0 = 0 \wedge (x)_1 \neq 0) \vee$$
$$\vee ((x)_1 = suc((x)_0)) \vee \exists z(((x)_0, z) \in y \wedge (z, (x)_1) \in y))]$$

Thanks For Your Attention!