Proof Surgery

George Metcalfe

Mathematical Institute, University of Bern

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The Proof System SL

identity axioms

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Soundness, Completeness, and Cut-Elimination

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where cuts are 'pushed upwards' in derivations until they vanish . . .

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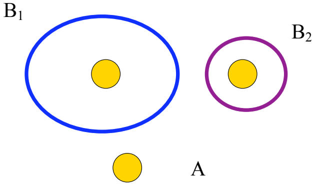
(1) establishing the **amalgamation property** via **Craig interpolation**

(2) establishing **densifiability** via **density elimination**.

Case Study (1): Amalgamation and Interpolation

Does some **class of algebras** $\mathcal{K}$ have the **amalgamation property**?



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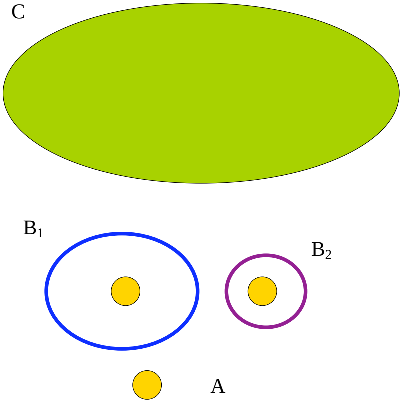

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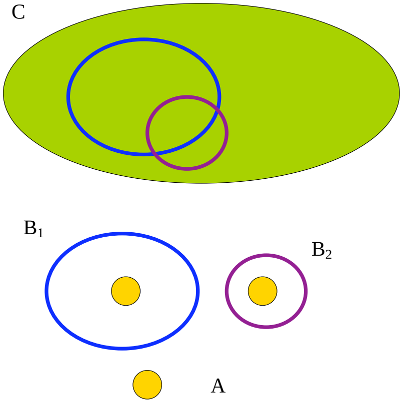

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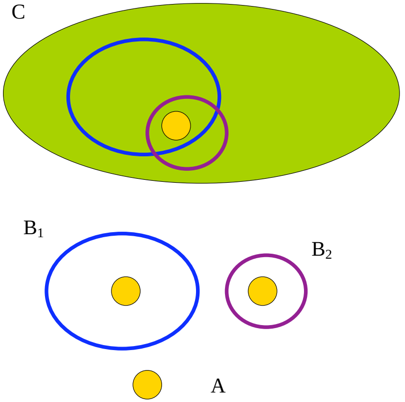

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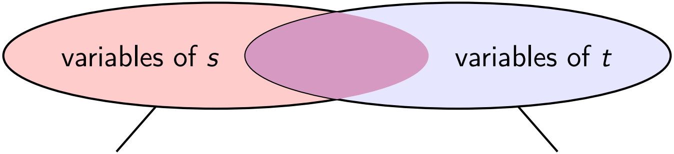

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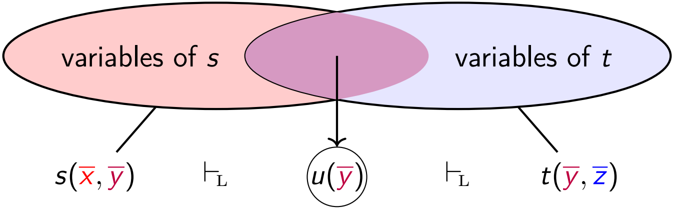

A Bridge Theorem

$\mathcal{K}$ has the **amalgamation property** $\iff$ L admits **interpolation**

Commutative Residuated Lattices

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The class $\mathcal{CRL}$ of all CRLs forms a **variety**; that is, it can be defined by a (finite) set of equations — or equivalently, it is closed under taking homomorphic images, subalgebras, and products.

Substructural Logics

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- the Full Lambek calculus.

Also covered by this framework are well-studied ordered algebras such as lattice-ordered groups and residuated lattices of ideals of rings.

The Amalgamation Property

A variety $\mathcal{V}$ has the **amalgamation property** if for any $\mathbf{A}, \mathbf{B}_1, \mathbf{B}_2 \in \mathcal{V}$ and embeddings $i\colon \mathbf{A} \to \mathbf{B}_1$ and $j\colon \mathbf{A} \to \mathbf{B}_2$,

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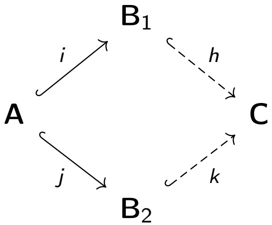

Our Question

Does $\mathcal{CRL}$ have the amalgamation property?

Craig Interpolation Property

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Theorem

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We establish the Craig interpolation property — and hence also the amalgamation property — for $\mathcal{CRL}$ by performing proof surgery on derivations in a suitable **sequent calculus**.

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A Sequent Calculus for Lattices

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cut rule

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A Sequent Calculus SCRL for $\mathcal{CRL}$

identity axioms

$$\frac{}{t \Rightarrow t}\ (\text{id})$$

cut rule

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An Example Derivation

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\dfrac{\dfrac{\dfrac{}{(x \to y) \lor (x \to z), x \Rightarrow y \lor z}\;(\lor\Rightarrow)}{(x \to y) \lor (x \to z) \Rightarrow x \to (y \lor z)}\;(\Rightarrow\to)}{\Rightarrow ((x \to y) \lor (x \to z)) \to (x \to (y \lor z))}\;(\Rightarrow\to)
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It follows directly that the equational theory of $\mathcal{CRL}$ is **decidable**.

Craig Interpolation for $\mathcal{CRL}$

Theorem

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Proof sketch. We prove that if $\vdash_{\text{SCRL}-(\textbf{cut})} \Gamma(\overline{x}, \overline{y}), \Pi(\overline{y}, \overline{z}) \Rightarrow t(\overline{y}, \overline{z})$,

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$$\Gamma_2, s \Rightarrow i_2 \quad \text{and} \quad \Pi_2, i_2 \Rightarrow t.$$

Defining $i := i_1 \to i_2$, we obtain derivations in $\mathrm{SCRL} - (\mathrm{cut})$

$$\dfrac{\dfrac{\dfrac{\vdots}{\Gamma_1, i_1 \Rightarrow t'} \quad \dfrac{\vdots}{\Gamma_2, s \Rightarrow i_2}}{\Gamma_1, \Gamma_2, t' \to s, i_1 \Rightarrow i_2}\,(\to\Rightarrow)}{\Gamma_1, \Gamma_2, t' \to s \Rightarrow i_1 \to i_2}\,(\Rightarrow\to) \qquad \dfrac{\dfrac{\vdots}{\Pi_1 \Rightarrow i_1} \quad \dfrac{\vdots}{\Pi_2, i_2 \Rightarrow t}}{\Pi_1, \Pi_2, i_1 \to i_2 \Rightarrow t.}\,(\to\Rightarrow)$$

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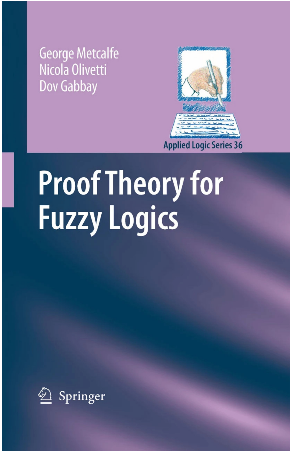
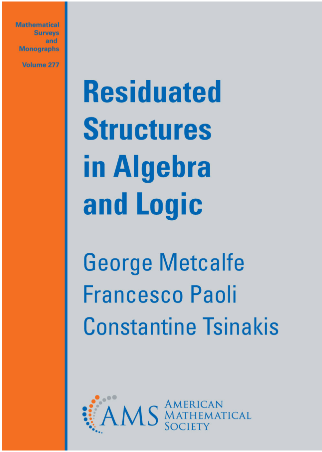
This method can be used to establish the amalgamation property for many other varieties of CRLs; algebraic methods can also be used in many cases (in particular, to establish failure) but no algebraic proof is known for $\mathcal{CRL}$.

An Ad Break

For more on residuated lattices and substructural logics, consult . . .





Case Study (2): Densifiability and Density Elimination

When do the chains of a variety $\mathcal{V}$ embed into dense chains in $\mathcal{V}$?

Chains and Dense Chains

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Proving that a semilinear variety of CRLs is densifiable is the crucial step for establishing **standard completeness** of a corresponding (fuzzy) logic, i.e., completeness with respect to algebras with lattice reduct $\langle [0,1], \min, \max \rangle$.

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But is $\mathcal{S}\text{em}\mathcal{CRL}$ densifiable? That is, can we prove that

$$\mathcal{CRL}^{\mathrm{c}} \models s \approx t \quad \iff \quad \mathcal{CRL}^{\mathrm{d}} \models s \approx t\,?$$

Two Approaches

Semantically . . .

Prove directly that each $\mathbf{A} \in \mathcal{CRL}^{\mathsf{c}}$ embeds into some $\mathbf{B} \in \mathcal{CRL}^{\mathsf{d}}$.

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Prove directly that each $\mathbf{A} \in \mathcal{CRL}^{\mathsf{c}}$ embeds into some $\mathbf{B} \in \mathcal{CRL}^{\mathsf{d}}$.

Syntactically . . .

Prove that every derivation in some proof system for $\mathcal{CRL}^{\mathsf{d}}$ can be transformed into a derivation in some proof system for $\mathcal{CRL}^{\mathsf{c}}$.

Hypersequents

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We denote arbitrary hypersequents by $\mathcal{G}, \mathcal{H}, \ldots$ and ignore brackets.

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$$\frac{\Pi \Rightarrow t \qquad \Gamma, s \Rightarrow u}{\Gamma, \Pi, t \to s \Rightarrow u}\ ({\to}{\Rightarrow}) \qquad \frac{\Gamma, s \Rightarrow t}{\Gamma \Rightarrow s \to t}\ ({\Rightarrow}{\to})$$

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- and the **communication** rule

$$\frac{\mathcal{G} \mid \Gamma_1, \Pi_1 \Rightarrow t_1 \quad \mathcal{G} \mid \Gamma_2, \Pi_2 \Rightarrow t_2}{\mathcal{G} \mid \Gamma_1, \Gamma_2 \Rightarrow t_1 \mid \Pi_1, \Pi_2 \Rightarrow t_2}\ (\mathsf{com})$$

An Example Derivation in $\mathrm{SCRL^c}$

$$\dfrac{}{\Rightarrow (x \to y) \vee (y \to x)}\ \text{(ec)}$$

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$$
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$$
\dfrac{\dfrac{}{\Rightarrow x \to y \mid\ \Rightarrow (x \to y) \vee (y \to x)}\ (\Rightarrow\vee)_2}{\dfrac{\Rightarrow (x \to y) \vee (y \to x) \mid\ \Rightarrow (x \to y) \vee (y \to x)}{\Rightarrow (x \to y) \vee (y \to x)}\ (\mathrm{ec})}\ (\Rightarrow\vee)_1
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$$
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Moreover, this system admits **cut elimination**, yielding

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It does not follow, however, that the equational theory of $\mathcal{CRL}^{\mathsf{c}}$ (equivalently, $\mathcal{S}\mathrm{em}\mathcal{CRL}$) is **decidable** — this is an open problem!

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The Density Rule

Let $\mathrm{SCRL}^{\mathsf{d}}$ consist of $\mathrm{SCRL}^{\mathsf{c}}$ extended with

$$\frac{\mathcal{G} \mid \Gamma \Rightarrow x \mid x, \Pi \Rightarrow t}{\mathcal{G} \mid \Gamma, \Pi \Rightarrow t}\ \text{(density)}$$

where x does not occur in the conclusion.

Proof Surgery (1)

Suppose that we have a derivation ending with

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Clearly, we can just remove the application of (com). More generally, we can use (cut) and cut elimination to repair derivations . . .

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The result follows by considering $\mathcal{G} = \mathcal{H} = (\mathcal{H}' \mid \Gamma \Rightarrow x \mid \Pi, x \Rightarrow t)$.

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Moreover, the densifiability of the variety of 'involutive' semilinear CRLs is an open problem . . .

Coda: Dropping Commutativity

A **residuated lattice** (or **RL**) is an algebraic structure $\langle A, \wedge, \vee, \cdot, \backslash, /, \mathrm{e} \rangle$ such that $\langle A, \wedge, \vee \rangle$ is a lattice, $\langle A, \cdot, \mathrm{e} \rangle$ is a monoid, and for all $a, b, c \in A$,

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- there is a hypersequent calculus for $\mathcal{RL}^{\mathrm{c}}$, but the variety of semilinear RLs is not densifiable — obtaining an equational axiomatization for the variety generated by $\mathcal{RL}^{\mathrm{d}}$ is an open problem!

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- However, we would also like to obtain general methods for using proof surgery to establish algebraic properties of ordered algebras, and understand these proof-theoretic methods algebraically.

Closing Credits

For further details and references, consult . . .