Logical Foundations of Categorization Theory

Scuola di Logica – Gargnano

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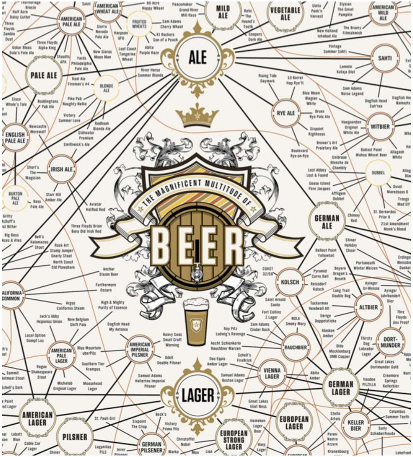
What is categorization?

From Wikipedia:
Categorization is the process in which ideas and objects are recognized, differentiated, and understood.
Ideally, a category illuminates a relationship between the subjects and objects of knowledge.
Categorization is fundamental in language, prediction, inference, decision-making and in all kinds of environmental interaction.



Overview and General Motivation

- **Truly interdisciplinary:** philosophy, cognition, social/management science, linguistics, AI.
- rapid development, **different approaches**;
- emerging **unifying perspective:** categories are dynamic in their essence; they shape and are shaped by processes of social interaction.
- **Data-driven** developments, both empirical and theoretical.
- However, what is **lacking**:
  - a **common ground** for the various approaches;
  - formal models addressing **dynamics** and connections with the processes of **social interaction**.
- **Research program:** logic as common ground; dynamics as starting point rather than outcome; systematic connection between dynamics and processes of social interaction.
- This involves **exploring seriously uncharted territory** in the mathematical theory of nonclassical logics.

How did I get interested in categories?

Uniform duality-theoretic approaches to nonclassical logics

- canonical extensions;
- unified correspondence;
- updates on algebras;
- multi-type calculi;
- multi-type **algebraic proof theory**.

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Mathematical theory of LE-logics (LE: lattice expansions)

- algebraic and state-based (aka relational) semantics;
- generalized Sahlqvist correspondence and canonicity;
- syntactic and semantic cut elimination, finite model property;
- Goldblatt-Thomason theorem.

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Can we make intuitive sense of LE-logics?

Basic lattice logic & main ideas

Language: $\mathcal{L} \ni \phi ::= p \in Prop \mid \top \mid \bot \mid \phi \wedge \phi \mid \phi \vee \phi$

Lattice Logic: Set of $\mathcal{L}$-sequents $\phi \vdash \psi$

- containing:

  $p \vdash p \quad \bot \vdash p \quad p \vdash \top \quad p \vdash p \vee q \quad q \vdash p \vee q \quad p \wedge q \vdash p \quad p \wedge q \vdash q$

- closed under:

  $\frac{\phi \vdash \chi \quad \chi \vdash \psi}{\phi \vdash \psi} \quad \frac{\phi \vdash \psi}{\phi(\chi/p) \vdash \psi(\chi/p)} \quad \frac{\chi \vdash \phi \quad \chi \vdash \psi}{\chi \vdash \phi \wedge \psi} \quad \frac{\phi \vdash \chi \quad \psi \vdash \chi}{\phi \vee \psi \vdash \chi}$

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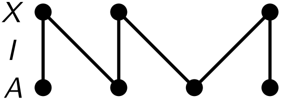
Challenge: Interpreting $\vee$ as 'or' and $\wedge$ as 'and' does not work, since 'and' and 'or' distribute over each other, while $\wedge$ and $\vee$ don't.
Proposal: Interpreting $\phi \in \mathcal{L}$ as **other entities than sentences**?
Examples: categories, concepts, theories, interrogative agendas.
The interpretation of $\vee$ and $\wedge$ in all these contexts is ok with failure of distributivity!
Approach:

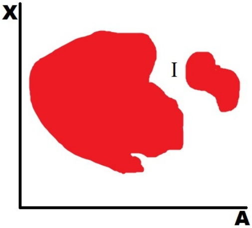
- Understand LE-logics as the logics of **these entities**;
- integrate LE-logics into more expressive logics capturing how these entities **interact** (e.g. with sentences, actions etc.).

Polarity-based semantics of LE-logics

Formal contexts (A, X, I) are abstract representations of databases:



A: set of *Objects*
X: set of *Features*
$I \subseteq A \times X$. Intuitively, aIx reads: object a has feature x



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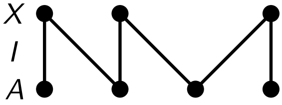

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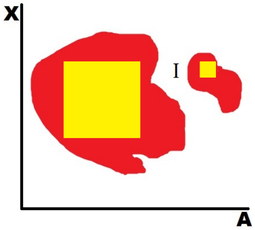

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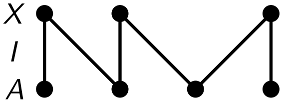

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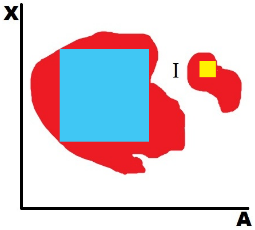

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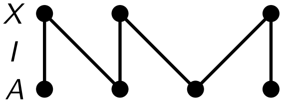

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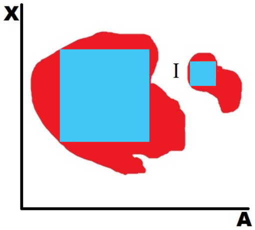
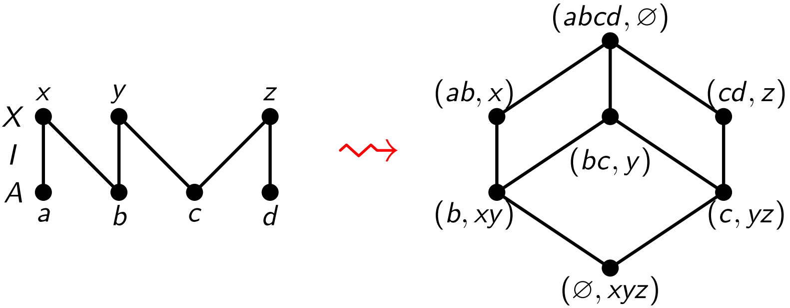


Formal concepts: "rectangles" maximally contained in I

Complex algebras



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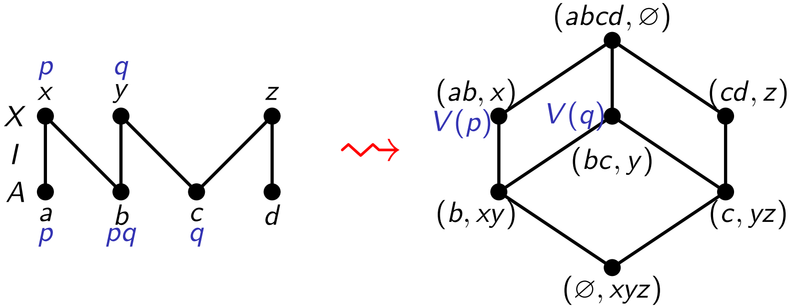
Formal contexts as $\mathcal{L}$-models



Let $\mathbb{P} = (A, X, I)$ and $\mathbb{P}^+$ be the complex algebra of $\mathbb{P}$.
Models: $\mathbb{M} := (\mathbb{P}, V)$ with $V : Prop \to \mathbb{P}^+$

$$V(p) := ([\![p]\!], (\![p]\!))$$

membership: $\mathbb{M}, a \Vdash p$ iff $a \in [\![p]\!]_{\mathbb{M}}$
description: $\mathbb{M}, x \succ p$ iff $x \in (\![p]\!)_{\mathbb{M}}$

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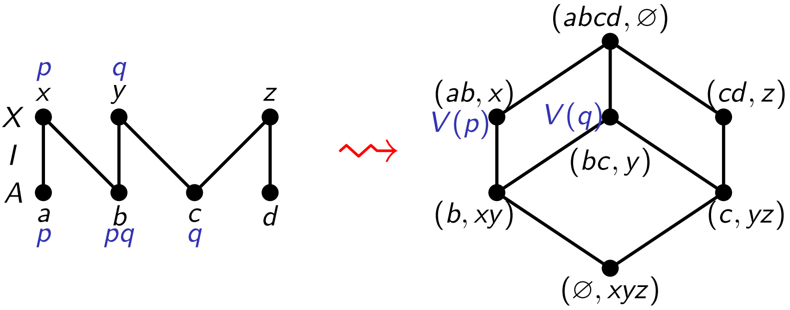
Formal contexts as $\mathcal{L}$-models



$\mathbb{M}, a \Vdash \bot$	iff	$\forall x(aIx)$		$\mathbb{M}, x \succ \bot$		always
$\mathbb{M}, a \Vdash \top$		always		$\mathbb{M}, x \succ \top$	iff	$\forall a(aIx)$

$\mathbb{M}, a \Vdash \phi \wedge \psi$ iff $\mathbb{M}, a \Vdash \phi$ and $\mathbb{M}, a \Vdash \psi$

$\mathbb{M}, x \succ \phi \wedge \psi$ iff for all $a \in A$, if $\mathbb{M}, a \Vdash \phi \wedge \psi$, then aIx

$\mathbb{M}, a \Vdash \phi \vee \psi$ iff for all $x \in X$, if $\mathbb{M}, x \succ \phi \vee \psi$, then aIx

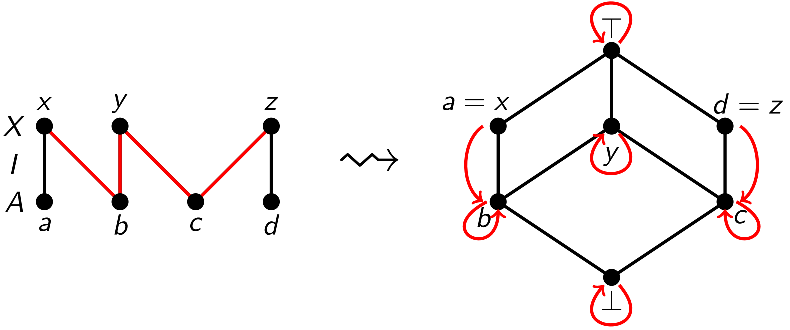
$\mathbb{M}, x \succ \phi \vee \psi$ iff $\mathbb{M}, x \succ \phi$ and $\mathbb{M}, x \succ \psi$

$$\mathbb{M} \models \phi \vdash \psi \text{ iff } [\![\phi]\!] \subseteq [\![\psi]\!] \text{ iff } (\![\psi]\!) \subseteq (\![\phi]\!)$$

Expanding the language with modal operators

Enriched formal contexts: $\mathbb{F} = (A, X, I, \{R_i \mid i \in Agents\})$

$R_i \subseteq A \times X$ and $\forall a((R^{\uparrow}[a])^{\downarrow\uparrow} = R^{\uparrow}[a])$ and $\forall x((R^{\downarrow}[x])^{\uparrow\downarrow} = R^{\downarrow}[x])$



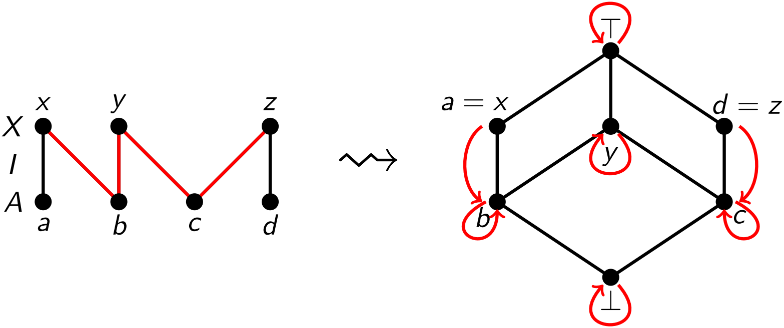
Language: $\mathcal{L}' \ni \phi ::= p \in Prop \mid \top \mid \bot \mid \phi \wedge \phi \mid \phi \vee \phi \mid \Box_i \phi$

$\Box_i \phi$: concept ϕ **according to** agent i

Logic:

- Additional axioms: $\top \vdash \Box_i \top \quad \Box_i \phi \wedge \Box_i \psi \vdash \Box_i(\phi \wedge \psi)$
- Additional rule: $\frac{\phi \vdash \psi}{\Box_i \phi \vdash \Box_i \psi}$

Interpretation of $\Box_i$-formulas on enriched formal contexts



$$V(\Box_i\phi) = \Box_i V(\phi) = (R_i^{\downarrow}[(\![\phi]\!)], (R_i^{\downarrow}[(\![\phi]\!)])^{\uparrow})$$

$\mathbb{M}, a \Vdash \Box_i\phi$ iff for all $x \in X$, if $\mathbb{M}, x \succ \phi$, then aR_ix

$\mathbb{M}, x \succ \Box_i\phi$ iff for all $a \in A$, if $\mathbb{M}, a \Vdash \Box_i\phi$, then aIx

Epistemic interpretation

Reflexivity aka Factivity

$$\forall p[\Box_i p \leq p]$$

iff $\forall a[a \in R_i^{\downarrow}[a^{\uparrow}]]$

iff $R_i \subseteq I$ Agent i's attributions are factually correct!

Transitivity aka Positive introspection

$$\forall p[\Box_i p \leq \Box_i \Box_i p]$$

iff $\forall x[R_i^{\downarrow}[x] \subseteq R_i^{\downarrow}[(R_i^{\downarrow}[x])^{\uparrow}]]$

iff $R_i \subseteq R_i ; R_i$

If agent i recognizes object a as an x-object, then i must also attribute to a all the features shared by x-objects according to i.

Non epistemic interpretation: rough concepts

Conceptual approximation spaces: $\mathbb{F} = (A, X, I, R_\Box, R_\Diamond)$ with $R_\Box \subseteq A \times X$ and $R_\Diamond \subseteq X \times A$, I-compatible and s.t. $R_\blacksquare \,;\, R_\Box \subseteq I$.

Fact: $\mathbb{F} \models \Box p \vdash \Diamond p$ iff $R_\blacksquare \,;\, R_\Box \subseteq I$

$$
\begin{array}{lll}
\mathbb{M}, a \Vdash \Box\varphi & \text{iff} & \text{for all } x \in X, \text{ if } \mathbb{M}, x \succ \varphi, \text{ then } aR_\Box x \\
\mathbb{M}, x \succ \Box\varphi & \text{iff} & \text{for all } a \in A, \text{ if } \mathbb{M}, a \Vdash \Box(\varphi), \text{ then } aIx, \\
\mathbb{M}, a \Vdash \Diamond\phi & \text{iff} & \text{for all } x \in X, \text{ if } \mathbb{M}, x \succ \Diamond\phi, \text{ then } aIx \\
\mathbb{M}, x \succ \Diamond\phi & \text{iff} & \text{for all } a \in A, \text{ if } \mathbb{M}, a \Vdash \phi, \text{ then } aR_\Diamond x.
\end{array}
$$

If (A, X, I) database and $R \subseteq A \times X$ I-compatible,

aIx stands for "object a has feature x"
aRx stands for "object a demonstrably has feature x"

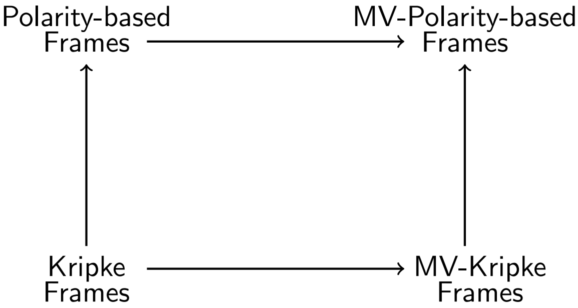
If $R_\Box := R$ and $R_\Diamond := R^{-1}$, then

$[\![\Box\phi]\!] = \{a \in A \mid \forall x(x \succ \phi \Rightarrow aRx)\}$ **certified members** of ϕ.

$$(\![\Diamond\phi]\!) = \{x \in X \mid \forall a(a \Vdash \phi \Rightarrow aRx)\},$$

hence $[\![\Diamond\phi]\!] :=$ **possible members** of ϕ.

From unified correspondence to parametric correspondence



- These (and other) semantic contexts relate to each other via embeddings;
- How do the f.o.-correspondents of (modal) axioms systematically relate to each other along these embeddings?
- Can we retrieve the intuitive meaning of these axioms in each context?